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Errors in the Use of Nonuniform Mesh Systems*

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The hypothesis that using nonsquare and nonuniform grids, with greatest density in the region of maximum change, produces minimum overall error in finite-difference solutions of a system of two nonlinear coupled second-order differential equations is examined. The model problem is Poiseuille pipe flow. The hypothesis is found through numerical experimentation to be false. Furthermore, it is found that maximum accuracy and minimum computation time are obtained through the use of an optimal sequence of iteration parameters in the alternating direction–iteration solution sequence with double-precision calculation on a square grid with 0.1 spacing.

INTRODUCTION

Numerical studies of fluid-dynamics problems are quite often concerned with flow around solid bodies where fairly large velocity gradients are encountered. In the vicinity of the body, it is often convenient to use a mesh system which is smaller than the mesh system imposed over most of the flow field and which might even be a nonsquare mesh system. Examples of this approach are seen in the studies of Whitaker and Wendel [1] and Thoman and Szewcyzk [2]. This study is concerned with a comparison of the numerical error that arises in the solution of the Navier–Stokes equations when a nonsquare, nonuniform mesh system is used. This type of problem is of interest because the velocity (or vorticity or stream function) can change rather sharply and the effect of the change is more pronounced in certain regions than in other regions. Therefore, it is felt that by increasing the density of the mesh points in the regions of greatest change, a marked improvement in the overall accuracy of the solution could be effected without the expense of increasing the density of the grid system everywhere. We offer the following study to show that such generalizations are not necessarily correct.

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THE PROBLEM

The problem chosen for study is that of Poiseuille flow in a pipe for which there is a well-known exact solution (see Schlichting [3]). This example arose during an investigation to determine numerically the onset of turbulence in Poiseuille pipe flow. The mathematical problem consisted of axisymmetrically perturbing the Poiseuille velocity distribution at some point in the pipe (see Crowder and Dalton [4]). Since the perturbation is expected to generate significant gradients of the vorticity and the stream function, it was felt desirable to use a denser mesh system in the vicinity of the perturbation as well as on the boundaries. The denser mesh system should allow for a more accurate representation of both the vorticity and the stream function. In checking the calculation procedure for the perturbed flow, it was noticed that varying numerical errors were obtained for the unperturbed solution when various nonuniform, nonsquare grid representations were used.

Investigation of the literature for nonuniform mesh systems showed the following: For a uniform mesh system, Young [5] gives the error term for a second partial derivative as behaving like $(h^2/4)\, \partial^4 U/\partial x^4$, where h is the uniform mesh spacing in the x direction. Young then gives a difference representation for the same second derivative over a nonuniform mesh system, i.e., when the mesh spacing goes like h on one side of the grid point in question and like sh $(0 < s \leqslant 1)$ on the other side. However, no error term is given for the nonuniform mesh case. Analysis of the error involved yields that the error term goes like $(1 - s)(h/3)\, \partial^3 U/\partial x^3$. Hence, a lower-order error in h is introduced for a nonuniform mesh system than for a uniform mesh system. It is also noted that for some physical problems $\partial^3 U/\partial x^3$ is significantly greater than $\partial^4 U/\partial x^4$. Hence, we choose to solve the problem of unperturbed Poiseuille pipe flow in order to obtain comparisons of the effect of the grid system on the accuracy of the finite-difference approximations to the Navier–Stokes equations.

GOVERNING DIFFERENTIAL EQUATIONS

A nondimensional axisymmetric form of the Navier–Stokes equations for viscous, incompressible flow in a circular pipe is

$$G_t - F_z \left(\frac{G}{r}\right)_r + \frac{F_r G_z}{r} = \frac{1}{R}\left[\left(\frac{1}{r}(rG)_r\right)_r + G_{zz}\right] \tag{1}$$

and

$$F_{zz} + r\left(\frac{F_r}{r}\right)_r = -rG, \tag{2}$$

where G is the vorticity, F is the stream function, and R is the Reynolds number. The Reynolds number is a parameter and is given by

$$R = \frac{\overline{W}D}{\nu},$$

where $\overline{W}$ is the average axial velocity component, D is the pipe diameter, and ν is the kinematic viscosity of the fluid.

The boundary conditions for the above differential equations are

$$G = 2r, \qquad F_z = 0 \qquad \text{on} \quad z = 0, \tag{3a}$$

$$G_z = F_{zz} = 0 \qquad \text{on} \quad z = L, \tag{3b}$$

$$G = F = 0 \qquad \text{on} \quad r = 0, \tag{3c}$$

$$F_r = 0, \qquad F = 0.25 \qquad \text{on} \quad r = 1, \tag{3d}$$

where $z = 0$ is the upstream boundary, $z = L$ is the downstream boundary, $r = 0$ is the centerline of the pipe, and $r = 1$ is the wall of the pipe. The initial conditions of the problem are given by

$$F = \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4 \qquad \text{for} \quad \begin{cases} 0 \leqslant z \leqslant L, \\ 0 \leqslant r \leqslant 1, \end{cases} \tag{4a}$$

and

$$G = 2r. \tag{4b}$$

With the above boundary and initial conditions, Eqs. (3) and (4), Eqs. (1) and (2) admit to an analytic solution,

$$F = \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4 \tag{5a}$$

and

$$G = 2r. \tag{5b}$$

This analytic solution is the steady flow solution to the above problem. Since we know this solution, we can use it as a comparison to determine the accuracy of the numerical procedures used to solve the mixed boundary–initial value problem.

For a derivation of the governing equations, see Schlichting [3]. For a discussion of the boundary conditions see Crowder and Dalton [4].

DIFFERENCE EQUATIONS

The solution of the differential equations is obtained by the use of the method of finite differences. For this technique, a net of grid points is introduced onto the region upon which the solution is to be found. We have chosen not to use equally spaced grid points, and therefore any function Φ is given by

$$\Phi(r, z, t) = \Phi(r_i, z_j, t_n) = \Phi^n_{i,j},$$

where

$$r_i = \sum_{m=1}^{i} \Delta r_m, \qquad \Delta r_1 = 0, \qquad 1 \leqslant i \leqslant I_{\max} \qquad (0 \leqslant r_i \leqslant 1), \tag{6a}$$

$$z_j = \sum_{m=1}^{j} \Delta z_m, \qquad \Delta z_1 = 0, \qquad 1 \leqslant j \leqslant J_{\max} \qquad (0 \leqslant z_j \leqslant L), \tag{6b}$$

and

$$t_n = \sum_{m=0}^{n} \Delta t_m, \qquad \Delta t_0 = 0, \qquad \Delta t_{2m+1} = \Delta t_{2m} \qquad (0 \leqslant t_n \leqslant T). \tag{6c}$$

For the above grid, central differences in space are used while forward differences in time are used. The operators which we will use are defined as follows:

$$\begin{aligned}\Phi_z = (\Phi^n_{i,j})_z = {} & \frac{1}{\Delta z_j + \Delta z_{j+1}} \Big[-\frac{\Delta z_{j+1}}{\Delta z_j} \Phi^n_{i,j-1} \\ & + \left(\frac{\Delta z_{j+1}}{\Delta z_j} - \frac{\Delta z_j}{\Delta z_{j+1}} \right) \Phi^n_{i,j} + \frac{\Delta z_j}{\Delta z_{j+1}} \Phi^n_{i,j+1} \Big] \\ & - \frac{\Delta z_j \, \Delta z_{j+1}}{3!} \Phi_{zzz}(r_i, \xi, t_n) \qquad (z_{j-1} \leqslant \xi \leqslant z_{j+1}),\end{aligned} \tag{7a}$$

$$\begin{aligned}\Phi_{zz} = (\Phi^n_{i,j})_{zz} = {} & \frac{2}{\Delta z_j + \Delta z_{j+1}} \Big[\frac{1}{\Delta z_j} \Phi^n_{i,j-1} \\ & - \left(\frac{1}{\Delta z_j} + \frac{1}{\Delta z_{j+1}} \right) \Phi^n_{i,j} + \frac{1}{\Delta z_{j+1}} \Phi^n_{i,j+1} \Big] \\ & + \frac{(\Delta z_j - \Delta z_{j+1})}{3} \Phi_{zzz}(r_i, z_j, t_n) \\ & - \frac{1}{12} (\Delta z_j^2 - \Delta z_j \, \Delta z_{j+1} + \Delta z_{j+1}^2) \Phi_{zzzz}(r_i, \xi, t_n) \\ & \qquad (z_{j-1} \leqslant \xi \leqslant z_{j+1}),\end{aligned} \tag{7b}$$

$$\Phi_r = (\Phi^n_{i,j})_r = \frac{1}{\Delta r_i + \Delta r_{i+1}} \Big[\frac{\Delta r_{i+1}}{\Delta r_i}\, \Phi^n_{i-1,j} + \Big(\frac{\Delta r_{i+1}}{\Delta r_i} - \frac{\Delta r_i}{\Delta r_{i+1}}\Big)\, \Phi^n_{i,j} + \frac{\Delta r_i}{\Delta r_{i+j}}\, \Phi^n_{i+1,j}\Big] - \frac{\Delta r_i\, \Delta r_{i+1}}{3!}\, \Phi_{rrr}(\xi, z_j, t_n) \qquad (r_{i-1} \leqslant \xi \leqslant r_{i+1}), \tag{7c}$$

$$\Phi_{rr} = (\Phi^n_{i,j})_{rr} = \frac{2}{\Delta r_i + \Delta r_{i+1}} \Big[\frac{1}{\Delta r_i}\, \Phi^n_{i-1,j} - \Big(\frac{1}{\Delta r_i} + \frac{1}{\Delta r_{i+1}}\Big)\, \Phi_{i,j} + \frac{1}{\Delta r_{i+1}}\, \Phi^n_{i+1,j}\Big] + \frac{(\Delta r_i - \Delta r_{i+1})}{3}\, \Phi_{rrr}(r_i, z_j, t_n) - \frac{1}{12}(\Delta r_i{}^2 - \Delta r_i\, \Delta r_{i+1} + \Delta r^2_{i+1})\, \Phi_{rrrr}(\xi, z_j, t_n) \qquad (r_{i-1} \leqslant \xi \leqslant r_{i+1}), \tag{7d}$$

and

$$\Phi_t = (\Phi^n_{i,j})_t = \frac{1}{\Delta t_n}\, [\Phi^{n+1}_{i,j} - \Phi^n_{i,j}] - \frac{\Delta t_n}{2}\, \Phi_{tt}(r_i, z_j, \xi) \qquad (t_n \leqslant \xi \leqslant t_{n+1}). \tag{7e}$$

Using Eq. (7) allows Eq. (2) to be approximated by

$$\rho_k F^{n+1,k+1}_{i,j} - (F^{n+1}_{i,j})^{k+1}_{rr} + \frac{1}{r_i}\, (F^{n+1}_{i,j})^{k+1}_r = \rho_k F^{n+1,k}_{i,j} + (F^{n+1}_{i,j})^k_{zz} \tag{8a}$$

and

$$\rho_k F^{n+1,k+2}_{i,j} - (F^{n+1}_{i,j})^{k+2}_{zz} = \rho_k F^{n+1,k+1}_{i,j} + (F^{n+1}_{i,j})^{k+1}_{rr} - \frac{1}{r_i}\, (F^{n+1}_{i,j})^{k+1}_r. \tag{8b}$$

The multipliers, ρ_k, in Eq. (8) are analogous to the Wachpress–Goode parameters. The Wachpress–Goode parameters are derived for Laplace's equation on a square region and are given by

$$\rho_k = b\, \Big(\frac{a}{b}\Big)^{(k+1)/(m+1)} \qquad k = 1, 2, \ldots, m, \tag{9}$$

where a and b are, respectively, the maximum and minimum eigenvalues of the operator matrix associated with the difference equations which approximate the Laplacian differential equation on a square, equally spaced grid (see Varga [6]). Since our differential equation is not the Laplacian and we allow our grid to be unequally spaced but rectangular over the axisymmetric flow field, then no analytic derivation of the values which ρ_k should have is available. Nevertheless, if we use the above formulation for the ρ_k for our problem, we find that the maximum and minimum eigenvalues of the operator matrix for the radial derivatives are always larger and smaller, respectively, than those for the operator matrix for the axial derivatives. Therefore, a and b are taken as the maximum and minimum eigenvalues of the operator matrix for the radial derivatives. Numerical experimentation also showed with this choice of a and b that use of five parameters for the iteration was optimal. The actual values of the ρ_k used in the calculations are given in Table 1.

TABLE 1

ρ_k for Use in Equation (8)

Grid	ρ_k
I	14.4554279116205
	45.5195753385026
	143.339356791502
	451.370010651717
	1421.34645414991
II and III	14.4366669994227
	46.5707761839117
	150.231157542023
	484.625822152668
	1563.33873305511
IV and VI	14.6371115664442
	46.5190382231239
	147.844805813021
	469.874000856377
	1493.33333333333
V	14.5031251532618
	33.1157808959826
	75.6150783201431
	172.656054444876
	394.235036169115

Again, using Eq. (7) allows Eq. (1) to be approximated by

$$\left[\frac{1}{\Delta t_{n+1}} + \frac{1}{r_i^2}(F_{i,j}^{n+1})_z + \frac{1}{r_i^2 R}\right] G_{i,j}^{n+1}$$

$$- \left[\frac{1}{r_i}(F_{i,j}^{n+1})_z + \frac{1}{r_i R}\right](G_{i,j}^{n+1})_r - \frac{1}{R}(G_{i,j}^{n+1})_{rr}$$

$$= \frac{1}{\Delta t_{n+1}} G_{i,j}^n - \frac{1}{r_i}(F_{i,j}^n)_r (G_{i,j}^n)_z + \frac{1}{R}(G_{i,j}^n)_{zz} \tag{10a}$$

and

$$\frac{1}{\Delta t_{n+2}} G_{i,j}^{n+2} + \frac{1}{r_i}(F_{i,j}^{n+2})_r (G_{i,j}^{n+2})_z - \frac{1}{R}(G_{i,j}^{n+2})_{zz}$$

$$= \left[\frac{1}{\Delta t_{n+2}} - \frac{1}{r_i^2}(F_{i,j}^{n+1})_z - \frac{1}{r_i^2 R}\right] G_{i,j}^{n+1}$$

$$+ \left[\frac{1}{r_i}(F_{i,j}^{n+1})_z + \frac{1}{r_i R}\right](G_{i,j}^{n+1})_r + \frac{1}{R}(G_{i,j}^{n+1})_{rr}, \tag{10b}$$

where we allow the time step, Δt_m, to vary from step to step in the solution process subject to the restriction

$$\Delta t_{2m+1} = \Delta t_{2m+2}. \tag{11}$$

The time step is used as a parameter to speed the iteration process and to insure convergence. The time step is selected such that the average number of iterations for the solutions at the $(2m+1)$-st and $(2m+2)$-nd times to converge is kept near a minimum.

For the derivatives involved in the boundary conditions, we use the differentiated Lagrangian interpolation formulas to obtain their finite-difference representation

$$G_{i,1}^n = 2r_i, \tag{12a}$$

$$(F_{i,1}^n)_z = 0, \tag{12b}$$

$$(G_{i,J_{\max}}^n)_z = 0, \tag{12c}$$

$$(F_{i,J_{\max}}^n)_{zz} = 0, \tag{12d}$$

$$G^n_{1,j} = 0, \tag{12e}$$

$$F^n_{1,j} = 0, \tag{12f}$$

$$G^n_{I_{\max},j} = -(F^n_{I_{\max},j})_{rr} + (F^n_{I_{\max},j})_r\,, \tag{12g}$$

$$F^n_{I_{\max},j} = 0.25 \tag{12h}$$

and

$$(F^n_{I_{\max},j})_r = 0. \tag{12i}$$

The initial conditions are given by

$$G^0_{i,j} = 2r_i \tag{13a}$$

and

$$F^0_{i,j} = \tfrac{1}{2}r_i^2 - \tfrac{1}{4}r_i^4. \tag{13b}$$

The difference equations, Eqs. (8) and (10), are solved on various grids, specified later, subject to the boundary and initial conditions, Eqs. (12) and (13). The method of solution used is an adaptation of the alternating direction implicit method of Peaceman and Rachford given by Young [5].

Before explaining the calculation procedure, we set forward the convergence tests that are applied. For the stream function we used

$$| F^{n,l,k+2m}_{i,j} - F^{n,l,k}_{i,j} | \leqslant \epsilon_f[\max_{i,j} | F^{n,l,k+2m-1}_{i,j} |], \tag{14}$$

where m is the number of ρ values used in the iterative procedure and k is the stream-function iteration counter. We chose ϵ_f such that $\epsilon_f \geqslant 1.0 \times 10^{-5}$.

The tests for convergence of the vorticity are

$$| G^{n,l+1}_{i,j} - G^{n,l}_{i,j} | \leqslant \epsilon_{g_i}[\max_{i,j} | G^{n,l}_{i,j} |] \; i \neq I_{\max}\,, \tag{15a}$$

and

$$| G^{n,l+1}_{I_{\max},j} - G^{n,l}_{I_{\max},j} | \leqslant \epsilon_{g_b}[\max_{j} | G^{n,l}_{I_{\max},j} |]. \tag{15b}$$

In Eqs. (14) and (15) l is the vorticity iteration counter. Equation (15a) was used where $i < I_{\max}$ and Eq. (15b) was used when $i = I_{\max}$. We demanded that $\epsilon_{g_b} = 2\epsilon_{g_i}$, and that $\epsilon_{g_i} \geqslant \epsilon_f$ at all iterations.

The solution to the difference equations is accomplished iteratively by first advancing the vorticity using Eqs. (10a) and (10b) for alternate time steps. Then Eq. (8) is iterated to convergence and the vorticity is recalculated on the basis of the updated stream function. This sequence is continued until the vorticity converges. Then the process is begun again with the alternate equation (10a) and/or (10b) for the next time step.

Grid System

The notation used for specifying a grid, (a, b, c), means that, starting at position a, increment by c until b is reached. The grids which we will compare are the following:

Grid I $\quad r_i = (0.0, 0.5, 0.1), (0.5, 0.7, 0.05), (0.7, 0.9, 0.1), (0.9, 1.0, 0.05),$
$\quad z_j = (0.0, 1.9, 0.1), (1.9, 2.1, 0.05), (2.1, 5.0, 0.1);$

Grid II $\quad r_i = (0.0, 0.5, 0.1), (0.5, 1.0, 0.05),$
$\quad z_j = (0.0, 1.9, 0.1), (1.9, 2.1, 0.05), (2.1, 5.0, 0.1);$

Grid III $\quad r_i = (0.0, 0.5, 0.1), (0.5, 1.0, 0.05),$
$\quad z_j = (0.0, 0.2, 0.05), (0.2, 1.9, 0.1), (1.9, 2.1, 0.05), (2.1, 4.8, 0.1), (4.8, 5.0, 0.05);$

Grid IV $\quad r_i = (0.0, 1.0, 0.05),$
$\quad z_j = (0.0, 5.0, 0.1);$

Grid V $\quad r_i = (0.0, 1.0, 0.1),$
$\quad z_j = (0.0, 5.0, 0.1);$

Grid VI $\quad r_i = (0.0, 1.0, 0.05),$
$\quad z_j = (0.0, 5.0, 0.05).$

Grids I, II, and III were used because they give a dense grid system in the region of the disturbance and/or the flow-field boundaries. Grid IV gives an additional density in the radial direction, which is the direction of most uncertainty in the difference approximations.

Grids V and VI furnish comparisons of the results on square grids as well as an estimate of the grid-size convergence.

The problem which was of major interest to the authors was the stability of Poiseuille pipe flow. For this problem it seemed desirable to have a denser mesh system in the vicinity of the point at which the disturbance is applied ($r = 0.6$, $z = 2.0$). Therefore, this is the common property of the nonuniform grids. In orde to cut computation time it is also desirable to keep the number of mesh points to a minimum, so the grids are made as sparse as possible.

The experimental error analysis of the solution of the difference equations on the above grids which we present is shown for both single- and double-precision calculations of the solution for each grid. We define the errors $e(F)$ and $e(G)$ at any grid point by

$$e(F_{i,j}) = (F_{i,j} - \tilde{F}_i)/\tilde{F}_i \,, \tag{16a}$$

$$e(G_{i,j}) = (G_{i,j} - \tilde{G}_i)/\tilde{G}_i \,, \tag{16b}$$

where $\tilde{F}$ and $\tilde{G}$ are true solutions given by Eq. (5) and F and G are calculated from Eqs. (8) and (10) with boundary and initial conditions, Eqs. (12) and (13).

The relevant quantities for comparison are

$$\overline{e(F)} = \frac{1}{N} \sum_i \sum_j e(F_{i,j}), \tag{17a}$$

$$\overline{e(G)} = \frac{1}{N} \sum_i \sum_j e(G_{i,j}), \tag{17b}$$

$$\overline{e_I(G)} = \frac{1}{N_1} \sum_{i=1}^{I_{\max}-1} \sum_j e(G_{i,j}), \tag{17c}$$

and

$$\overline{e_B(G)} = \frac{1}{J_{\max}} \sum_j e(G_{I_{\max},j}), \tag{17d}$$

where $N = (I_{\max})(J_{\max})$ and $N_1 = N - J_{\max}$. The absence of limits on the summation implies summing over the range of the index.

RESULTS

The solution to Eqs. (8) and (10) was obtained at four consecutive steps for each grid system. The errors, as specified by Eq. (17), at each of these time steps are presented in Tables 2–5, (the second number in each column is the power of 10 which should multiply the first number in order to obtain the true value), which show that Grid II is the best of the nonuniform Grids I, II, and III and that Grid I is the worst mesh system tested based on a comparison of the errors. The mesh systems with uniform grid spacing, Grids IV, V, and VI, are all significantly better than Grids I, II, and III. The grid system with 0.05 square spacing,

TABLE 2

Mean Relative Error, Time Step 1, Time 0.01

Grid system	$\overline{e(F)}$	$\overline{e(G)}$	$\overline{e_I(G)}$	$\overline{e_B(G)}$
I S[a]	1.815 −3	−1.041 −2	−4.859 −5	− .1451
I D[b]	1.819 −3	−1.045 −2	−4.815 −5	− .1457
II S	3.240 −3	−4.216 −3	−1.996 −5	−6.716 −2
II D	3.245 −3	−4.272 −3	−1.953 −5	−6.807 −2
III S	3.240 −3	−4.227 −3	−2.002 −5	−6.734 −2
III D	3.245 −3	−4.278 −3	−1.955 −5	−6.816 −2
IV S	−1.232 −5	4.385 −5	−4.288 −7	9.293 −4
IV D	−5.564 −15	1.601 −14	−6.432 −17	3.376 −13
V S	−3.211 −6	1.930 −6	−6.908 −7	−1.432 −5
V D	−1.739 −15	−2.086 −14	−1.010 −16	−2.193 −14
VI S	−1.492 −5	4.991 −5	−4.278 −7	1.057 −3
VI D	−4.248 −15	1.562 −14	−7.876 −17	3.297 −13

[a] S indicates single-precision calculation.
[b] D indicates double-precision calculation.

TABLE 3

Mean Relative Error, Time Step 2, Time 0.02

Grid system	$\overline{e(F)}$	$\overline{e(G)}$	$\overline{e_I(G)}$	$\overline{e_B(G)}$
I S[a]	1.808 −3	−1.015 −2	−9.736 −5	− .1408
I D[b]	1.813 −3	−1.020 −2	−9.651 −5	− .1415
II S	3.236 −3	−4.103 −3	−3.993 −5	−6.505 −2
II D	3.242 −3	−4.157 −3	−3.914 −5	−6.592 −2
III S	3.236 −3	−4.113 −3	−4.007 −5	−6.520 −2
III D	3.242 −3	−4.162 −3	−3.924 −5	−6.601 −2
IV S	−1.242 −5	4.308 −5	−9.236 −7	9.232 −4
IV D	−5.504 −15	1.595 −14	−1.083 −16	3.372 −13
V S	−3.584 −6	−5.065 −7	−1.244 −6	6.872 −6
V D	−1.744 −15	8.332 −16	−2.201 −16	1.137 −14
VI S	−1.484 −5	4.593 −5	8.741 −7	9.819 −4
VI D	−4.281 −15	1.510 −14	−1.498 −16	3.201 −13

[a] S indicates single-precision calculation.
[b] D indicates double-precision calculation.

TABLE 4

Mean Relative Error, Time Step 3, Time 0.031

Grid system	$\overline{e(F)}$	$\overline{e(G)}$	$\overline{e_I(G)}$	$\overline{e_B(G)}$
I S[a]	1.801 −3	−1.017 −2	−1.488 −4	− .1404
I D[b]	1.806 −3	−1.007 −2	−1.467 −4	− .1391
II S	3.233 −3	−4.057 −3	−6.076 −5	−6.400 −2
II D	3.238 −3	−4.099 −3	−5.948 −5	−6.470 −2
III S	3.233 −3	−4.065 −3	−6.100 −5	−6.412 −2
III D	3.238 −3	−4.105 −3	−5.966 −5	−6.478 −2
IV S	−1.308 −5	4.197 −5	−1.438 −6	9.102 −4
IV D	−5.545 −15	1.596 −14	−1.608 −16	3.384 −13
V S	−4.646 −6	−6.380 −7	−1.882 −6	1.180 −5
V D	−1.765 −15	5.457 −16	−3.315 −16	9.317 −15
VI S	−1.493 −5	4.540 −5	−1.310 −6	9.796 −4
VI D	−4.374 −15	1.395 −14	−2.305 −16	2.975 −13

[a] S indicates single-precision calculation.
[b] D indicates double-precision calculation.

TABLE 5

Mean Relative Error, Time Step 4, Time 0.042

Grid system	$\overline{e(F)}$	$\overline{e(G)}$	$\overline{e_I(G)}$	$\overline{e_B(G)}$
I S[a]	1.794 −3	−9.901 −3	−2.003 −4	− .1360
I D[b]	1.800 −3	−9.942 −3	−1.971 −4	− .1366
II S	3.230 −3	−3.993 −3	−8.159 −5	−6.266 −2
II D	3.235 −3	−4.042 −3	−7.989 −5	−6.347 −2
III S	3.229 −3	−4.067 −3	−8.187 −5	−6.385 −2
III D	3.235 −3	−4.047 −3	−8.012 −5	−6.355 −2
IV S	−1.359 −5	4.088 −5	−1.921 −6	8.968 −4
IV D	−5.559 −15	1.570 −14	−1.889 −16	3.335 −13
V S	−4.543 −6	−1.130 −6	−2.455 −6	3.698 −5
V D	−1.781 −15	−4.780 −16	−4.236 −16	9.493 −15
VI S	−1.534 −5	4.914 −5	−1.723 −6	1.066 −3
VI D	−4.359 −15	1.467 −14	−3.110 −16	3.142 −13

[a] S indicates single-precision calculation.
[b] D indicates double-precision calculation.

Grid VI, gives a better representation of the stream function and boundary vorticity than was obtained from Grid IV, the uniformly spaced, nonsquare mesh system. However, Grid IV gives a better representation of the interior vorticity than does Grid VI. The grid system with 0.1 square spacing, Grid V, has less error for all times shown, as seen in Tables 2–5; this is true for both single and double precision. The overall vorticity values are seen to have the least error for both single- and double-precision computation on Grid V for the three greatest time values as seen in Tables 3–5. Grid VI has the least error for both single- and double-precision calculation of the interior vorticity for all time levels shown. However, the error in boundary vorticity for Grid VI is greater than that for Grid V. Therefore, we rank the grid systems in order of increasing preference from the standpoint of errors produced in the solutions as follows: I, III, II, IV, VI, and V. Also due to the magnitude of the change in the errors, it is evident that use of double-precision calculations for the non-uniform grids, Grids I, II, and III, effects no appreciable improvement in the accuracy of the solution. For the uniform grids, Grids IV, V, and VI, double-precision calculations show an improvement consistent with the increased number of available digits for the computation.

The increase in the error of the boundary vorticity over the interior vorticity obtained in all of the solutions is expected. Since the value of the boundary vorticity is calculated by differencing the stream function, any errors in the stream function are magnified in the value obtained for the boundary vorticity. This magnification is an inverse function of step size in the radial direction. Therefore, the errors in the values of the boundary vorticity are seen to be consistent with the grid system used.

The results which are presented in Tables 2–5 could be expanded to show individual errors at grid points, or to show errors for each grid line, but this is unnecessary since the values presented are indicative of the results everywhere.

The reason that the interior mean vorticity increases with time is that the errors in the boundary vorticity are spread inward slowly as the solution progresses. For time step 1 the points at which the vorticity is significantly in error are on the boundary only. At time step 4, the next two interior grid lines also show noticeable error in the vorticity. Table 6 shows the computing time expended to calculate the solutions for the four time steps presented in Tables 2–5.

The sequence of times obtained for either single- or double-precision calculations,

TABLE 6

Computation Time in Minutes

Grid	I	II	III	IV	V	VI
Single precision	1.16	1.38	1.51	1.48	.95	2.79
Double precision	1.22	1.54	1.60	1.72	.83	3.32

using the same convergence criteria, is as expected since the time required to compute the solution increases with an increasing number of grid points. However, when we compare the times involved between single- and double-precision calculations, we note that double-precision calculation takes less time for Grid V than does single-precision. This is the reverse of what happens for all the other grids. Therefore, from the standpoint of computation time involved in the calculation, the grids are ranked in increasing order of preference as VI, III, IV, II, I, and V for single-precision and as VI, IV, III, II, I, and V for double-precision calculation. Hence, the preferred grid system from this standpoint is Grid V with double-precision calculations.

SYMMARY

On the basis of the results obtained, we conclude that calculation of the stream function and vorticity for Poiseuille pipe flow is most accurately and efficiently accomplished using double-precision calculations on Grid V, a square, equally spaced mesh system of 0.1 grid size. This result is in direct contrast with the original hypothesis set out in the introduction, i.e., that nonuniform and nonsquare grids should allow an increased accuracy in the representation of the vorticity and stream function.

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